Déterminants 3×3

Algèbre linéaire et géométrie vectorielle (201-NYC)
Session H2025 — Yannick Delbecque

Notation 1. *Si* $\vec{u} = (u_1, u_2, u_3)$*, alors*

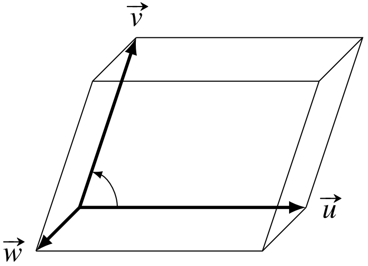
$$\begin{pmatrix} \text{——} & \vec{u} & \text{——} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$$

et aussi

$$\begin{vmatrix} \text{——} & \vec{u} & \text{——} \\ \text{——} & \vec{v} & \text{——} \\ \text{——} & \vec{w} & \text{——} \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

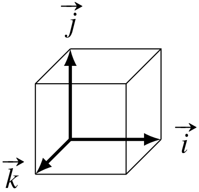
Définition 1 (Déterminant 3×3)**.**
Le volume du parallélipipède engendré par $\vec{u}$, $\vec{v}$ *et* $\vec{w}$ *est dénoté par*

$$\begin{vmatrix} — & \vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$



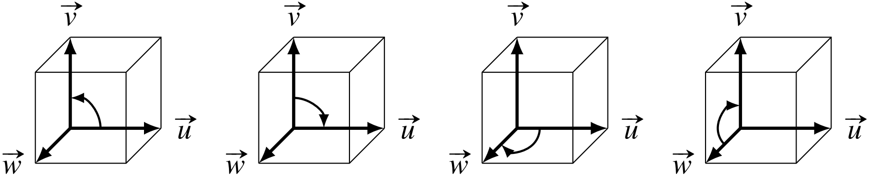
Hypothèse 1 (Propriétés des déterminants)**.**

$$(D1)\quad \begin{vmatrix} — & \vec{i} & — \\ — & \vec{j} & — \\ — & \vec{k} & — \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

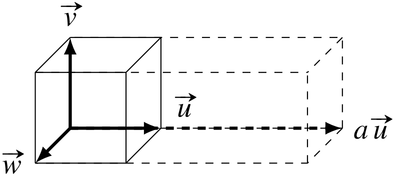


(D2) Échanger deux lignes change le signe

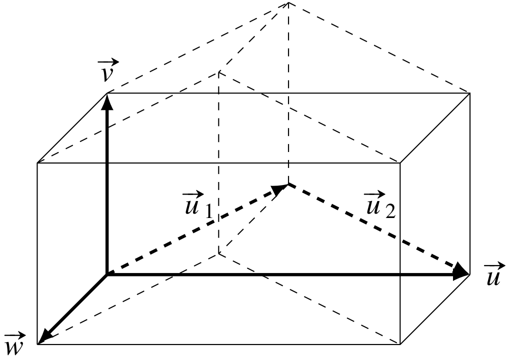
$$\begin{vmatrix} — & \vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = -\begin{vmatrix} — & \vec{v} & — \\ — & \vec{u} & — \\ — & \vec{w} & — \end{vmatrix} = -\begin{vmatrix} — & \vec{u} & — \\ — & \vec{w} & — \\ — & \vec{v} & — \end{vmatrix} = -\begin{vmatrix} — & \vec{w} & — \\ — & \vec{v} & — \\ — & \vec{u} & — \end{vmatrix}$$



$$(D3)\quad \begin{vmatrix} — & a\vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = a \begin{vmatrix} — & \vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix}$$



$$(D4)\quad \begin{vmatrix} — & \vec{u}_1 + \vec{u}_2 & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = \begin{vmatrix} — & \vec{u}_1 & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} + \begin{vmatrix} — & \vec{u}_2 & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix}$$



Lemme 1.

$$\begin{vmatrix} — & a\vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = \begin{vmatrix} — & \vec{u} & — \\ — & a\vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = \begin{vmatrix} — & \vec{u} & — \\ — & \vec{v} & — \\ — & a\vec{w} & — \end{vmatrix} = a\begin{vmatrix} — & \vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix}$$

Lemme 2.

$$\begin{vmatrix} \text{---} & \vec{u}_1 + \vec{u}_2 & \text{---} \\ \text{---} & \vec{v} & \text{---} \\ \text{---} & \vec{w} & \text{---} \end{vmatrix} = \begin{vmatrix} \text{---} & \vec{u}_1 & \text{---} \\ \text{---} & \vec{v} & \text{---} \\ \text{---} & \vec{w} & \text{---} \end{vmatrix} + \begin{vmatrix} \text{---} & \vec{u}_2 & \text{---} \\ \text{---} & \vec{v} & \text{---} \\ \text{---} & \vec{w} & \text{---} \end{vmatrix}$$

$$\begin{vmatrix} \text{---} & \vec{u} & \text{---} \\ \text{---} & \vec{v}_1 + \vec{v}_2 & \text{---} \\ \text{---} & \vec{w} & \text{---} \end{vmatrix} = \begin{vmatrix} \text{---} & \vec{u} & \text{---} \\ \text{---} & \vec{v}_1 & \text{---} \\ \text{---} & \vec{w} & \text{---} \end{vmatrix} + \begin{vmatrix} \text{---} & \vec{u} & \text{---} \\ \text{---} & \vec{v}_2 & \text{---} \\ \text{---} & \vec{w} & \text{---} \end{vmatrix}$$

$$\begin{vmatrix} \text{---} & \vec{u} & \text{---} \\ \text{---} & \vec{v} & \text{---} \\ \text{---} & \vec{w}_1 + \vec{w}_2 & \text{---} \end{vmatrix} = \begin{vmatrix} \text{---} & \vec{u} & \text{---} \\ \text{---} & \vec{v} & \text{---} \\ \text{---} & \vec{w}_1 & \text{---} \end{vmatrix} + \begin{vmatrix} \text{---} & \vec{u} & \text{---} \\ \text{---} & \vec{v} & \text{---} \\ \text{---} & \vec{w}_2 & \text{---} \end{vmatrix}$$

Lemme 3.

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = -\begin{vmatrix} u_2 & u_1 & u_3 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} = -\begin{vmatrix} u_1 & u_3 & u_2 \\ v_1 & v_3 & v_2 \\ w_1 & w_3 & w_2 \end{vmatrix} = -\begin{vmatrix} u_3 & u_2 & u_1 \\ v_3 & v_2 & v_1 \\ w_3 & w_2 & w_1 \end{vmatrix}$$

Théorème 1 (Développement déterminant 3×3)**.**

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

Démonstration.

$$
\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\overset{(D4)}{=}
\begin{vmatrix} u_1 & 0 & 0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
+
\begin{vmatrix} 0 & u_2 & 0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
+
\begin{vmatrix} 0 & 0 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
$$

$$
=
\begin{vmatrix} u_1 & 0 & 0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
-
\begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix}
+
\begin{vmatrix} 0 & 0 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
$$

$$
=
\begin{vmatrix} u_1 & 0 & 0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
-
\begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix}
-
\begin{vmatrix} 0 & u_3 & 0 \\ v_1 & v_3 & v_2 \\ w_1 & w_3 & w_2 \end{vmatrix}
$$

$$
=
\begin{vmatrix} u_1 & 0 & 0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
-
\begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix}
+
\begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}
$$

$$= \begin{vmatrix} u_1 & 0 & 0 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & 0 & 0 \\ v_1 & 0 & 0 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & 0 & 0 \\ 0 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & 0 & 0 \\ \left(\frac{v_1}{u_1}\right) u_1 & 0 & 0 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & 0 & 0 \\ 0 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}$$

$$= \frac{v_1}{u_1} \begin{vmatrix} u_1 & 0 & 0 \\ u_1 & 0 & 0 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & 0 & 0 \\ 0 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & 0 & 0 \\ 0 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}$$

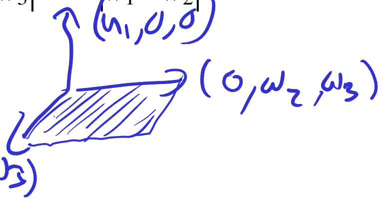
$$\cdots = \begin{vmatrix} u_1 & 0 & 0 \\ 0 & v_2 & v_3 \\ 0 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ v_2 & v_1 & v_3 \\ w_2 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ v_3 & v_1 & v_2 \\ w_3 & w_1 & w_2 \end{vmatrix}$$

$$\cdots = \begin{vmatrix} u_1 & 0 & 0 \\ 0 & v_2 & v_3 \\ 0 & w_2 & w_3 \end{vmatrix} - \begin{vmatrix} u_2 & 0 & 0 \\ 0 & v_1 & v_3 \\ 0 & w_1 & w_3 \end{vmatrix} + \begin{vmatrix} u_3 & 0 & 0 \\ 0 & v_1 & v_2 \\ 0 & w_1 & w_2 \end{vmatrix}$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \quad \square$$

hauteur

Base



Règle de Sarrus

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \left|\begin{array}{ccc|cc} u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 \\ w_1 & w_2 & w_3 & w_1 & w_2 \end{array}\right.$$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1 - u_1v_3w_2 - u_2v_1w_3$$

$$= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1$$

Les deux coïncident !

$$\begin{aligned}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1\end{aligned}$$

Proposition 1.

$$\begin{vmatrix} — & \vec{u} & — \\ — & \vec{v} & — \\ — & \vec{w} & — \end{vmatrix} = \begin{vmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{vmatrix}$$

Théorème 2. *Le système d'équations linéaires*

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

a une solution unique si et seulement si

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

Démonstration. On suppose que $a_1 \neq 0$.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

$$\sim \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_1a_2 & a_1b_2 & a_1c_2 & a_1d_2 \\ a_1a_3 & a_1b_3 & a_1c_3 & a_1d_3 \end{pmatrix}$$

$$\sim \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & a_1b_2 - a_2b_1 & a_1c_2 - a_2c_1 & a_1d_2 - a_2d_1 \\ 0 & a_1b_3 - a_3b_1 & a_1c_3 - a_3c_1 & a_1d_3 - a_3d_1 \end{pmatrix}$$

$$\sim \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & a_1b_2 - a_2b_1 & a_1c_2 - a_2c_1 & a_1d_2 - a_2d_1 \\ 0 & (a_1b_3 - a_3b_1)(a_1b_2 - a_2b_1) & (a_1c_3 - a_3c_1)(a_1b_2 - a_2b_1) & (a_1d_3 - a_3d_1)(a_1b_2 - a_2b_1) \end{pmatrix}$$

$$\sim \begin{pmatrix} a_1 & b_1 & c_1 & * \\ 0 & a_1b_2 - a_2b_1 & a_1c_2 - a_2c_1 & * \\ 0 & 0 & (a_1c_3 - a_3c_1)(a_1b_2 - a_2b_1) - (a_1c_2 - a_2c_1)(a_1b_3 - a_3b_1) & * \end{pmatrix}$$

$$
\begin{aligned}
(a_1c_3 - a_3c_1)(a_1b_2 - a_2b_1) - (a_1c_2 - a_2c_1)(a_1b_3 - a_3b_1)& \\
&= a_1^2b_2c_3 - a_1a_2b_1c_3 - a_1a_3b_2c_1 + a_2a_3b_1c_1 \\
&\qquad - a_1^2b_3c_2 + a_1a_3b_1c_2 + a_1a_2b_3c_1 - a_2a_3b_1c_1 \\
&= a_1^2b_2c_3 - a_1a_2b_1c_3 - a_1a_3b_2c_3 - a_1^2b_3c_2 + a_1a_3b_1c_2 + a_1a_2b_3c_1 \\
&= a_1\left(a_1b_2c_3 - a_2b_1c_3 - a_3b_2c_3 - a_1b_3c_2 + a_3b_1c_2 + a_2b_3c_1\right)
\end{aligned}
$$

Comme $a_1 \neq 0$, le système d'équations linéaires donné a une solution unique si et seulement si

$$
a_1b_2c_3 - a_2b_1c_1 - a_3b_2c_3 - a_1b_3c_2 + a_3b_1c_2 + a_2b_3c_1 \neq 0. \qquad \square
$$

Théorème 3 (Cramer)**.**

Si $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$*, alors le système d'équations linéaires*

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

a pour unique solution

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Théorème 4. *Les vecteurs u, v et w sont linéairement indépendants si et seulement si*

$$\begin{vmatrix} \text{——} & \overrightarrow{u} & \text{——} \\ \text{——} & \overrightarrow{v} & \text{——} \\ \text{——} & \overrightarrow{w} & \text{——} \end{vmatrix} \neq 0$$

Théorème 5. *Les points A, B, C et D sont coplanaires si et seulement si*

$$\begin{vmatrix} \text{---} & \overrightarrow{AB} & \text{---} \\ \text{---} & \overrightarrow{AC} & \text{---} \\ \text{---} & \overrightarrow{AD} & \text{---} \end{vmatrix} \neq 0$$

Exemple 1. *Déterminer si* $A = (1,2,-1)$*,* $B = (2,0,1)$ $C = (0,1,1)$ *et* $D = (1,1,1)$ *sont coplanaires.*

solution

$$\overrightarrow{AB} = (2,0,1) - (1,2,-1) = (1,-2,2)$$
$$\overrightarrow{AC} = (0,1,1) - (1,2,-1) = (-1,-1,2)$$
$$\overrightarrow{AD} = (1,1,1) - (1,2,-1) = (0,-1,2)$$

$$\begin{vmatrix} \text{——} & \overrightarrow{AB} & \text{——} \\ \text{——} & \overrightarrow{AC} & \text{——} \\ \text{——} & \overrightarrow{AD} & \text{——} \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & -1 & 2 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= (1)\begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} + (2)\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}$$

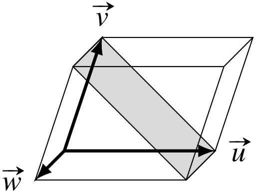
$$= (1)(0) - (-2)(-2) + (2)(1)$$

$$= -2$$

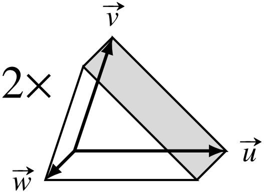
Comme le déterminant est différent de zéro, les vecteurs $\vec{u}$, $\vec{v}$ *et* $\vec{w}$ *ne sont pas coplanaires.*

Volumes pyramides et prismes

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

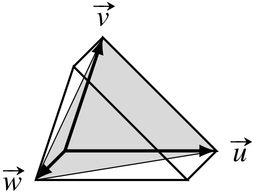


$= 2\times$

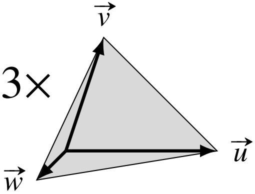


$$\frac{1}{2}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\frac{1}{2}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



$= 3\times$



$$\frac{1}{6}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$